Equivalence of Qudit Gate Operations

S. Balakrishnan

Abstract— The basic unit of quantum information content is qubit which is usually a two level quantum system. Among the various two qubit gates, the exchange gate, otherwise known as SWAP gate is very fundamental. Extension of quantum operations from two level (qubit) to any arbitrary dimensional level (qudit) is important but not straightforward. The circuit constructions of qudit SWAP operation using various qudit gates are known. In this work, equivalence of some important qudit operations is illustrated. Such equivalence may be exploited to minimize the circuit complexity in the previously proposed qudit SWAP gate constructions.

Index Terms— Circuit complexity, Circuit equivalence, Qubits, Qudits, SWAP gate

1 Introduction

Quantum bit, shortly known as qubit, is the basic unit of quantum information processing. Any two level quantum mechanical system can be regarded as a qubit. For instance, two level atoms, spin states of particles and polarizations of photon are all legitimate candidates for qubits [1]. Qubits are manipulated by quantum operations which are also known as qubit gates. Importantly, two qubit operations are fundamental in the construction of universal quantum circuits [2]. Some well known two qubit gates are Controlled NOT, shortly $CNOT$ and $SWAP$.

In general, quantum information processing is possible with arbitrary d dimensional Hilbert space associated with quantum systems. In such a case, basic information processing unit is qudit. Sometimes, it is worth studying the quantum operations for qudit systems. However, the extension of two-qubit operation to qudit operation is not always straightforward. A typical example is the implementation of $SWAP$ operation in the qudit setting.

In this work, we have shown the existence of equivalence between some of the qudit gates. These gates are the generalization of $CNOT$ operation at the qudit level. Notably, these qudit gates are proposed with the aim to implement qudit $SWAP$ operation [3], [4], [5], [6], [7]. The identified equivalence of qudit gates gains its significance in proposing new qudit $SWAP$ constructions.

2 Qudit Operations

We first introduce important two-qubit gates, namely $CNOT$ and $SWAP$. While the former gate performs the NOT operation depending upon the control qubit, the latter gate interchanges the input states. Notably, a sequence of three $CNOT$ gates can implement $SWAP$ operation [1]. One of the qubits of $CNOT$ gate is known as control and the other qubit is target. Whenever the control qubit is in $|1\rangle$ state, the target qubit changes its value from $|0\rangle$ to $|1\rangle$ or from $|1\rangle$ to $|0\rangle$. The $CNOT$ operation is given by

$$CNOT|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle. \quad (1)$$

where the symbol $\oplus$ represents the addition modulo two. The circuit symbol for $CNOT$ is shown in Fig. 1(a) where the dot represents the control and the symbol $\oplus$ represents the target. The $SWAP$ gate interchanges the input states, say $|\phi\rangle$ and $|\psi\rangle$. The action of $SWAP$ is given by

$$SWAP|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle. \quad (2)$$

The circuit symbol for two-qubit $SWAP$ gate is shown in Fig. 1(b). Note that the input states can be of d dimensional as well, in that case we have qudit $SWAP$ operation.

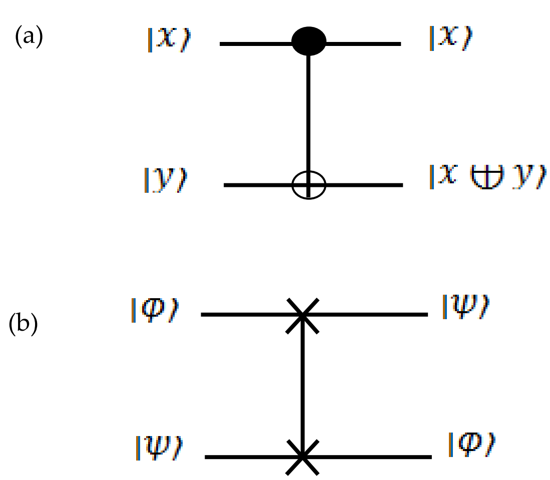

Fig. 1 Two-qubit gates. (a) $CNOT$ (b) $SWAP$.

Now we introduce some qudit gates which are the generalizations of qudit $CNOT$ operation. Among the various versions, CX_d is the widely used and its operations is given

- *S. Balakrishnan is currently Postdoctoral Fellow in the Materials Physics division in the School of Advanced Sciences at VIT University, Vellore 632014, Tamilnadu, India, Ph: +91 9944187604, E-mail: physicsbalki@gmail.com* by

$$CX_d|x\rangle|y\rangle = |x\rangle|x+y\rangle \tag{3}$$

with a modulo d addition. We consider addition and subtraction operations for modulo d. The inverse of above operation is

$$CX_d^{\dagger}|x\rangle|y\rangle = |x\rangle|y-x\rangle \tag{4}$$

with modulo d subtraction. It is worth noting that $CX_d \neq CX_d^{\dagger}$, implying the gates are not Hermitian. This is the main issue in the qudit generalization of $CNOT$. It is important to mention that both CX_d and $CX_d^{\dagger}$ are necessary to perform qudit $SWAP$ operation [3], [5], [6]. With the aim to have Hermitian qudit $CNOT$ operation, $GXOR$ is introduced [8]. The action of $GXOR$ is defined as

$$GXOR|x\rangle|y\rangle = |x\rangle|x-y\rangle. \tag{5}$$

A new Hermitian qudit operation, $C\ddot{X}$ is defined in [9]. The action of $C\ddot{X}$ gate is given below:

$$C\ddot{X}|x\rangle|y\rangle = |x\rangle|-x-y\rangle. \tag{6}$$

Note that a much simpler circuit for qudit $SWAP$ gate can be constructed using $C\ddot{X}$ gate [9]. Apart from these gates, we have

$$X_d|x\rangle = |d-x\rangle = |-x\rangle. \tag{7}$$

This gate provides modulo d complement of the input state.

3 Main result

The main result of this work is identifying the equivalence of the qudit gates introduced in the previous section. The equivalence can be checked by the action of the gates on the input states $|\phi\rangle$ and $|\psi\rangle$ of arbitrary dimension d. In the following, the subscripts i and j represent the control and target qudits respectively.

$$GXOR_{1,2} \equiv X_{d2}CX_{d1,2}^{\dagger} \tag{8}$$

$$GXOR_{1,2} \equiv CX_{d1,2}X_{d2} \tag{9}$$

$$GXOR_{2,1} \equiv X_{d1}CX_{d2,1}^{\dagger} \tag{10}$$

$$GXOR_{2,1} \equiv CX_{d2,1}X_{d1} \tag{11}$$

$$C\ddot{X}_{1,2} \equiv X_{d2}CX_{d1,2} \tag{12}$$

$$C\ddot{X}_{1,2} \equiv CX_{d1,2}^{\dagger}X_{d2} \tag{13}$$

$$C\ddot{X}_{2,1} \equiv X_{d1}CX_{d2,1} \tag{14}$$

$$C\ddot{X}_{2,1} \equiv CX_{d2,1}^{\dagger}X_{d1} \tag{15}$$

The proposed equivalence suggests that wherever the specific combinations of two qudit operations are present in a circuit that can be replaced by a single qudit operation. This indicates the possibility of reduction in the number of gates in a circuit. Further, using the equivalence given by (8) and (13), we can show that

$$GXOR_{1,2} \equiv X_{d2}C\ddot{X}_{1,2}X_{d2}. \tag{16}$$

Similarly, using the equivalence given by (11) and (14) we can show that

$$GXOR_{2,1} \equiv X_{d1}C\ddot{X}_{2,1}X_{d1}. \tag{17}$$

From (16) and (17), it is clear that the two Hermitian qudit gates are related in a simple fashion. Thus, these two gates can be used interchangeably in the circuits.

4 Conclusion

In this work, we have illustrated the existence of equivalence between various qudit gates and these gates are useful in the construction of qudit $SWAP$ circuit. The identification of equivalence of the qudit operations can be exploited to minimize the circuit complexity in the previously proposed qudit $SWAP$ gate constructions. Moreover, qudit $SWAP$ circuit with lesser number of gates can be proposed. The results of the work in this direction is under communication [10].

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